

On a q -analog of a Sahi result

Olga Bershtein

Institute for Low Temperature Physics and Engineering, Kharkov, Ukraine
e-mail: bershtein@ilt.kharkov.ua

Abstract

We obtain a q -analog of a well known Sahi result on the joint spectrum of $S(GL_n \times GL_n)$ -invariant differential operators with polynomial coefficients on the vector space of complex $n \times n$ -matrices.

Keywords: factorial Schur polynomials, Capelli identities, quantum groups, quantum prehomogeneous vector spaces.

MSC: 17B37, 20G42, 16S32.

1 Introduction

Start with recalling some well-known facts. Denote by Mat_n the vector space of complex $n \times n$ -matrices. The group $K = S(GL_n \times GL_n)$ acts on Mat_n by

$$(u, v)Z = uZv^{-1}, \quad (u, v) \in K, Z \in \text{Mat}_n.$$

This induces the natural K -actions in the vector spaces $\mathbb{C}[\text{Mat}_n]$ of holomorphic polynomials, $D[\text{Mat}_n]$ of differential operators with constant coefficients, and $\text{PD}[\text{Mat}_n]$ of differential operators with polynomial coefficients. The well-known Hua theorem claims that

$$\mathbb{C}[\text{Mat}_n] = \bigoplus_{\lambda \in \Lambda_n} \mathbb{C}[\text{Mat}_n]_\lambda,$$

where $\Lambda_n = \{\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{Z}_+^n \mid \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n\}$ and $\mathbb{C}[\text{Mat}_n]_\lambda$ is a simple finite dimensional K -module with the highest weight

$$(\lambda_1 - \lambda_2, \dots, \lambda_{n-1} - \lambda_n, 2\lambda_n, \lambda_{n-1} - \lambda_n, \dots, \lambda_1 - \lambda_2). \quad (1)$$

Similarly, $D[\text{Mat}_n] = \bigoplus_{\lambda \in \Lambda_n} D_\lambda$, with $D_\lambda \cong \mathbb{C}[\text{Mat}_n]_\lambda^*$ (we are using here the standard pairing between polynomials and differential operators with constant coefficients).

Let $y_\nu = \sum_i v_i w_i \in \text{PD}[\text{Mat}_n]$, where $\{v_i\}$ is a basis in $\mathbb{C}[\text{Mat}_n]_\nu$, and $\{w_i\}$ is the dual basis in D_ν . $y_\nu|_{\mathbb{C}[\text{Mat}_n]_\lambda}$ is a scalar operator since $\mathbb{C}[\text{Mat}_n]_\lambda$ is a simple K -module and y_ν is K -invariant. Sahi arranges studying an explicit formula for these scalars [11], [5, Proposition 3.3]:

$$y_\nu|_{\mathbb{C}[\text{Mat}_n]_\lambda} = \mathfrak{s}_\nu(\lambda_1 + n - 1, \lambda_2 + n - 2, \dots, \lambda_{n-1} + 1, \lambda_n), \quad (2)$$

where the *factorial* Schur polynomial \mathfrak{s}_ν associated to a partition $\nu = (\nu_1, \dots, \nu_n)$ is defined by

$$\mathfrak{s}_\nu(x_1, x_2, \dots, x_n) = \frac{\det \left(\prod_{m=0}^{\nu_j + n - j - 1} (x_i - m) \right)_{1 \leq i, j \leq n}}{\prod_{i < j} (x_i - x_j)},$$

(see [2, 9]).

This paper presents a q -analog of this formula.

2 The main statement

Let $q \in (0, 1)$. All algebras are assumed to be associative and unital, and \mathbb{C} is the ground field.

Recall that $U_q \mathfrak{sl}_{2n}$ is a Hopf algebra with generators $\{E_i, F_i, K_i, K_i^{-1}\}_{i=1}^{2n-1}$ and the Drinfeld-Jimbo relations [3]

$$\begin{aligned} K_i K_j &= K_j K_i, & K_i K_i^{-1} &= K_i^{-1} K_i = 1; \\ K_i E_i &= q^2 E_i K_i, & K_i F_i &= q^{-2} F_i K_i; \\ K_i E_j &= q^{-1} E_j K_i, & K_i F_j &= q F_j K_i, & |i - j| &= 1; \\ K_i E_j &= E_j K_i, & K_i F_j &= F_j K_i, & |i - j| &> 1; \\ E_i F_j - F_j E_i &= \delta_{ij} \frac{K_i - K_i^{-1}}{q - q^{-1}}; \\ E_i^2 E_j - (q + q^{-1}) E_i E_j E_i + E_j E_i^2 &= 0, & |i - j| &= 1; \\ F_i^2 F_j - (q + q^{-1}) F_i F_j F_i + F_j F_i^2 &= 0, & |i - j| &= 1; \\ E_i E_j - E_j E_i &= F_i F_j - F_j F_i = 0, & |i - j| &\neq 1. \end{aligned}$$

The coproduct, the counit, and the antipode are defined as follows:

$$\begin{aligned} \Delta E_j &= E_j \otimes 1 + K_j \otimes E_j, & \varepsilon(E_j) &= 0, & S(E_j) &= -K_j^{-1} E_j, \\ \Delta F_j &= F_j \otimes K_j^{-1} + 1 \otimes F_j, & \varepsilon(F_j) &= 0, & S(F_j) &= -F_j K_j, \\ \Delta K_j &= K_j \otimes K_j, & \varepsilon(K_j) &= 1, & S(K_j) &= K_j^{-1}, & j &= 1, \dots, 2n-1. \end{aligned}$$

Equip the Hopf algebra $U_q \mathfrak{sl}_{2n}$ with an involution $*$:

$$(K_j^{\pm 1})^* = K_j^{\pm 1}, \quad E_j^* = \begin{cases} K_j F_j, & j \neq n, \\ -K_j F_j, & j = n, \end{cases} \quad F_j^* = \begin{cases} E_j K_j^{-1}, & j \neq n, \\ -E_j K_j^{-1}, & j = n. \end{cases}$$

$U_q \mathfrak{su}_{n,n} \stackrel{\text{def}}{=} (U_q \mathfrak{sl}_{2n}, *)$ is a $*$ -Hopf algebra. Denote by $U_q \mathfrak{k} \subset U_q \mathfrak{sl}_{2n}$ the Hopf subalgebra generated by $E_j, F_j, j \neq n$, and $K_i, K_i^{-1}, i = 1, \dots, 2n-1$.

Introduce a $*$ -algebra $\text{Pol}(\text{Mat}_n)_q$, which is one of the basic objects in the theory of quantum bounded symmetric domains (see, for example, [13]). First, introduce a well-known quantum analog $\mathbb{C}[\text{Mat}_n]_q$ of the algebra $\mathbb{C}[\text{Mat}_n]$ of holomorphic polynomials on the matrix space (see, for example, [4], chap. 9.2). It is defined by the generators $z_a^\alpha, a, \alpha = 1, \dots, n$, and the following relations

$$z_a^\alpha z_b^\beta - q z_b^\beta z_a^\alpha = 0, \quad a = b \quad \& \quad \alpha < \beta, \quad \text{or} \quad a < b \quad \& \quad \alpha = \beta, \quad (3)$$

$$z_a^\alpha z_b^\beta - z_b^\beta z_a^\alpha = 0, \quad \alpha < \beta \quad \& \quad a > b, \quad (4)$$

$$z_a^\alpha z_b^\beta - z_b^\beta z_a^\alpha - (q - q^{-1}) z_a^\beta z_b^\alpha = 0, \quad \alpha < \beta \quad \& \quad a < b. \quad (5)$$

Similarly, denote by $\mathbb{C}[\overline{\text{Mat}}_n]_q$ an algebra with the generators $(z_a^\alpha)^*, a, \alpha = 1, \dots, n$ and the defining relations

$$(z_b^\beta)^* (z_a^\alpha)^* - q (z_a^\alpha)^* (z_b^\beta)^* = 0, \quad a = b \quad \& \quad \alpha < \beta, \quad \text{or} \quad a < b \quad \& \quad \alpha = \beta, \quad (6)$$

$$(z_b^\beta)^* (z_a^\alpha)^* - (z_a^\alpha)^* (z_b^\beta)^* = 0, \quad \alpha < \beta \quad \& \quad a > b, \quad (7)$$

$$(z_b^\beta)^* (z_a^\alpha)^* - (z_a^\alpha)^* (z_b^\beta)^* - (q - q^{-1}) (z_b^\alpha)^* (z_a^\beta)^* = 0, \quad \alpha < \beta \quad \& \quad a < b. \quad (8)$$

Let $\mathbb{C}[\text{Mat}_n \oplus \overline{\text{Mat}}_n]_q$ be an algebra with the generators $z_a^\alpha, (z_a^\alpha)^*, a, \alpha = 1, \dots, n$, and the defining relations (3) – (8) and

$$(z_b^\beta)^* z_a^\alpha = q^2 \sum_{a', b'=1}^n \sum_{\alpha', \beta'=1}^n R(b, a, b', a') R(\beta, \alpha, \beta', \alpha') z_{a'}^{\alpha'} \left(z_{b'}^{\beta'}\right)^* + (1 - q^2) \delta_{ab} \delta^{\alpha\beta}, \quad (9)$$

where $\delta_{ab}, \delta^{\alpha\beta}$ are Kronecker symbols,

$$R(j, i, j', i') = \begin{cases} q^{-1}, & i \neq j \text{ \& } j = j' \text{ \& } i = i', \\ 1, & i = j = i' = j', \\ -(q^{-2} - 1), & i = j \text{ \& } i' = j' \text{ \& } i' > i, \\ 0, & \text{otherwise.} \end{cases}$$

Finally, let $\text{Pol}(\text{Mat}_n)_q$ denotes the $*$ -algebra $(\mathbb{C}[\text{Mat}_n \oplus \overline{\text{Mat}}_n]_q, *)$ with the involution given by $*$: $z_a^\alpha \mapsto (z_a^\alpha)^*$.

It is very important for our purposes that $\text{Pol}(\text{Mat}_n)_q$ is a q -analog of the algebra of differential operators with polynomial coefficients $\text{PD}[\text{Mat}_n]$ mentioned in the Introduction. Indeed, the latter algebra is derivable from $\text{Pol}(\text{Mat}_n)_q$ via the change of generators $z_a^\alpha \rightarrow (1 - q^2)^{-1/2} z_a^\alpha$ and a subsequent formal passage to the limit as $q \rightarrow 1$.

$\text{Pol}[\text{Mat}_n]_q$ can be equipped with a $U_q \mathfrak{su}_{n,n}$ -module algebra structure via such formulas (see [13, Sec. 9,10]): for $a, \alpha = 1, \dots, n$

$$K_n^{\pm 1} z_a^\alpha = \begin{cases} q^{\pm 2} z_a^\alpha, & a = n \text{ \& } \alpha = n, \\ q^{\pm 1} z_a^\alpha, & a = n \text{ \& } \alpha \neq n \text{ or } a \neq n \text{ \& } \alpha = n, \\ z_a^\alpha, & \text{otherwise,} \end{cases}$$

$$F_n z_a^\alpha = q^{1/2} \cdot \begin{cases} 1, & a = n \text{ \& } \alpha = n, \\ 0, & \text{otherwise,} \end{cases} \quad E_n z_a^\alpha = -q^{1/2} \cdot \begin{cases} q^{-1} z_a^n z_n^\alpha, & a \neq n \text{ \& } \alpha \neq n, \\ (z_n^\alpha)^2, & a = n \text{ \& } \alpha = n, \\ z_n^n z_a^\alpha, & \text{otherwise,} \end{cases}$$

and for $k \neq n$

$$K_k^{\pm 1} z_a^\alpha = \begin{cases} q^{\pm 1} z_a^\alpha, & k < n \text{ \& } a = k \text{ or } k > n \text{ \& } \alpha = 2n - k, \\ q^{\mp 1} z_a^\alpha, & k < n \text{ \& } a = k + 1 \text{ or } k > n \text{ \& } \alpha = 2n - k + 1, \\ z_a^\alpha, & \text{otherwise,} \end{cases}$$

$$F_k z_a^\alpha = q^{1/2} \cdot \begin{cases} z_{a+1}^\alpha, & k < n \text{ \& } a = k, \\ z_a^{\alpha+1}, & k > n \text{ \& } \alpha = 2n - k, \\ 0, & \text{otherwise,} \end{cases}$$

$$E_k z_a^\alpha = q^{-1/2} \cdot \begin{cases} z_{a-1}^\alpha, & k < n \text{ \& } a = k + 1, \\ z_a^{\alpha-1}, & k > n \text{ \& } \alpha = 2n - k + 1, \\ 0, & \text{otherwise.} \end{cases}$$

In the sequel we are using standard results on finite dimensional $U_q \mathfrak{k}$ -modules of type **1**, see [3]. As a $U_q \mathfrak{k}$ -module, $\mathbb{C}[\text{Mat}_n]_q = \bigoplus_{\lambda \in \Lambda_n} \mathbb{C}[\text{Mat}_n]_{q,\lambda}$, with $\mathbb{C}[\text{Mat}_n]_{q,\lambda}$ being a simple finite dimensional $U_q \mathfrak{k}$ -module with the highest weight given by (1). Also, $\mathbb{C}[\overline{\text{Mat}}_n]_q = \bigoplus_{\lambda \in \Lambda_n} \mathbb{C}[\overline{\text{Mat}}_n]_{q,\lambda}$, with $\mathbb{C}[\overline{\text{Mat}}_n]_{q,\lambda} \approx \mathbb{C}[\text{Mat}_n]_{q,\lambda}^*$ as $U_q \mathfrak{k}$ -modules. Then

$$(\text{Pol}(\text{Mat}_n)_q)^{U_q \mathfrak{k}} = \bigoplus_{\nu \in \Lambda_n} \text{Pl}_\nu, \quad \text{Pl}_\nu = (\mathbb{C}[\text{Mat}_n]_{q,\nu} \otimes \mathbb{C}[\overline{\text{Mat}}_n]_{q,\nu})^{U_q \mathfrak{k}}, \quad \dim \text{Pl}_\nu = 1.$$

Let $\{v_j\} \subset \mathbb{C}[\text{Mat}_n]_{q,\nu}$ be a basis and $\{w_j\} \subset \mathbb{C}[\overline{\text{Mat}}_n]_{q,\nu}$ the dual basis. Then $\sum_j v_j w_j \in \text{Pl}_\nu$. Introduce q -minors

$$z^{\wedge k}_I \stackrel{\text{def}}{=} \sum_{s \in S_k} (-q)^{l(s)} z_{i_1}^{j_{s(1)}} z_{i_2}^{j_{s(2)}} \cdots z_{i_k}^{j_{s(k)}},$$

$$I = \{(i_1, i_2, \dots, i_k) | 1 \leq i_1 < i_2 < \dots < i_k \leq n\},$$

$$J = \{(j_1, j_2, \dots, j_k) | 1 \leq j_1 < j_2 < \dots < j_k \leq n\}.$$

It can be verified easily that

$$v_\nu = (z^{\wedge n}_{\{1, \dots, n\}})^{\nu_n} \prod_{k=1}^{n-1} (z^{\wedge k}_{\{1, \dots, k\}})^{\nu_k - \nu_{k+1}}$$

is a highest weight vector of $\mathbb{C}[\text{Mat}_n]_{q,\nu}$. Consider a basis $\{v_j\} \subset \mathbb{C}[\text{Mat}_n]_{q,\nu}$ that contains v_ν . The isomorphism between $\mathbb{C}[\overline{\text{Mat}}_n]_{q,\nu}$ and $\mathbb{C}[\text{Mat}_n]_{q,\nu}^*$ mentioned above can be chosen so that the dual basis $\{w_j\}$ contains v_ν^* . Introduce $y_\nu \in \text{Pl}_\nu$ by

$$y_\nu = \sum_j v_j w_j.$$

Denote by \mathcal{H} a $\text{Pol}(\text{Mat}_n)_q$ -module with a generator f_0 and defining relations

$$(z_a^\alpha)^* f_0 = 0, \quad a, \alpha = 1, \dots, n,$$

and by T_F the corresponding representation of $\text{Pol}(\text{Mat}_n)_q$ in \mathcal{H} . The statements of the following proposition are proved in [13].

Proposition 1 1. $\mathcal{H} = \mathbb{C}[\text{Mat}_n]_q f_0$.

2. \mathcal{H} is a simple $\text{Pol}(\text{Mat}_n)_q$ -module.

3. There exists a unique sesquilinear form (\cdot, \cdot) on \mathcal{H} with the following properties:

i) $(f_0, f_0) = 1$; ii) $(fv, w) = (v, f^*w)$ for all $v, w \in \mathcal{H}$, $f \in \text{Pol}(\text{Mat}_n)_q$.

4. The form (\cdot, \cdot) is positive definite.

5. T_F is a faithful representation.

So, \mathcal{H} is a pre-Hilbert space, and T_F is an irreducible faithful $*$ -representation.

Evidently, \mathcal{H} inherits the decomposition

$$\mathcal{H} = \bigoplus_{\lambda \in \Lambda_n} \mathcal{H}_\lambda. \tag{10}$$

Proposition 2 (*D. Shklyarov*) $y_\nu y_\lambda = y_\lambda y_\nu$ for all partitions $\nu \neq \lambda$.

Proof. Commutativity is deduced from the faithfulness of T_F and the simplicity of the summands in (10). \square

As in the classical case, $T_F(y_\nu)|_{\mathcal{H}_\lambda}$ are scalar operators for all ν and λ . The main goal of this paper is to obtain an explicit formula for the scalars $T_F(y_\nu)|_{\mathcal{H}_\lambda}$.

Recall the notation of the q -factorial Schur polynomials [5]: for $\nu \in \Lambda_n$

$$\mathfrak{s}_\nu(x_1, x_2, \dots, x_n; q) = \frac{\det \left(\prod_{m=0}^{\nu_j + n - j - 1} (x_i - q^m) \right)_{1 \leq i, j \leq n}}{\prod_{i < j} (x_i - x_j)}.$$

Theorem 1 For all partitions $\nu, \lambda \in \Lambda_n$

$$T_F(y_\nu)|_{\mathcal{H}_\lambda} = (-q)^{\sum_{i=1}^n \nu_i} q^{\text{const}} \mathfrak{s}_\nu(q^{2(\lambda_1+n-1)}, q^{2(\lambda_2+n-2)}, \dots, q^{2(\lambda_{n-1}+1)}, q^{2\lambda_n}; q^2)$$

with $\text{const} = -\sum_{i=1}^n \nu_i(\nu_i + 2n - 2i)$.

This theorem is a natural generalization of the following result. Let $\mathbf{1}^k \stackrel{\text{def}}{=} (\underbrace{1, \dots, 1}_k, 0, \dots, 0)$.

Theorem 2 [1, Theorem 1] For all $k = 1, 2, \dots, n$ and all $\lambda \in \Lambda_n$

$$T_F(y_{\mathbf{1}^k})|_{\mathcal{H}_\lambda} = (-q)^k q^{-k^2-2k(n-k)} \mathfrak{s}_{\mathbf{1}^k}(q^{2(\lambda_1+n-1)}, q^{2(\lambda_2+n-2)}, \dots, q^{2(\lambda_{n-1}+1)}, q^{2\lambda_n}; q^2).$$

First, we prove some auxiliary statements.

Lemma 1 The subalgebra $\text{Pol}(\text{Mat}_n)_q^{U_q \mathfrak{k}}$ is generated by the elements $y_{\mathbf{1}^k}$.

Proof. Equip the $U_q \mathfrak{k}$ -module $(\text{Pol}(\text{Mat}_n)_q)^{U_q \mathfrak{k}}$ with the natural grading

$$(\text{Pol}(\text{Mat}_n)_q)^{U_q \mathfrak{k}} = \bigoplus_{j=0}^{\infty} (\text{Pol}(\text{Mat}_n)_q)_j^{U_q \mathfrak{k}}, \quad (\text{Pol}(\text{Mat}_n)_q)_j^{U_q \mathfrak{k}} = \bigoplus_{\nu \in \Lambda_n, |\nu|=j} \text{Pl}_\nu.$$

Hence $\dim(\text{Pol}(\text{Mat}_n)_q)_j^{U_q \mathfrak{k}} = \#\{\nu \in \Lambda_n, |\nu| = j\}$.

It follows from Theorem 2 that monomials $\{y_{\mathbf{1}^1}^{a_1} y_{\mathbf{1}^2}^{a_2} \dots y_{\mathbf{1}^n}^{a_n}\}$ are linear independent. This fact allows one to denote by $\mathbb{C}[y_{\mathbf{1}^1}, y_{\mathbf{1}^2}, \dots, y_{\mathbf{1}^n}] \subset (\text{Pol}(\text{Mat}_n)_q)^{U_q \mathfrak{k}}$ the subalgebra in generated by the elements $y_{\mathbf{1}^1}, y_{\mathbf{1}^2}, \dots, y_{\mathbf{1}^n}$. It is easy to see that $\deg y_{\mathbf{1}^k} = k$. Hence $\mathbb{C}[y_{\mathbf{1}^1}, y_{\mathbf{1}^2}, \dots, y_{\mathbf{1}^n}] \cap (\text{Pol}(\text{Mat}_n)_q)_j^{U_q \mathfrak{k}}$ is a linear span of $\{y_{\mathbf{1}^1}^{a_1} y_{\mathbf{1}^2}^{a_2} \dots y_{\mathbf{1}^n}^{a_n} | a_1 + 2a_2 + \dots + na_n = j\}$, and

$$\dim(\mathbb{C}[y_{\mathbf{1}^1}, y_{\mathbf{1}^2}, \dots, y_{\mathbf{1}^n}] \cap (\text{Pol}(\text{Mat}_n)_q)_j^{U_q \mathfrak{k}}) = \#\{a_1, \dots, a_n \in \mathbb{Z}_+ | a_1 + 2a_2 + \dots + na_n = j\}.$$

So, $\dim(\mathbb{C}[y_{\mathbf{1}^1}, y_{\mathbf{1}^2}, \dots, y_{\mathbf{1}^n}] \cap (\text{Pol}(\text{Mat}_n)_q)_j^{U_q \mathfrak{k}}) = \dim(\text{Pol}(\text{Mat}_n)_q)_j^{U_q \mathfrak{k}}$, and $(\text{Pol}(\text{Mat}_n)_q)^{U_q \mathfrak{k}}$ is generated by $y_{\mathbf{1}^1}, y_{\mathbf{1}^2}, \dots, y_{\mathbf{1}^n}$. \square

The next statements concern symmetric polynomials and vanishing conditions in the spirit of papers [6, 10] (for their classical analogs, see [5, 7, 8]). Recall some notations from [6]. Fix non-zero real numbers q and t . For every $\lambda \in \Lambda_n$ we define $\bar{\lambda} = (q^{\lambda_1}, q^{\lambda_2} t^{-1}, \dots, q^{\lambda_n} t^{-n+1})$. We use the following short notation: $|\lambda| = \sum_{i=1}^n \lambda_i$ for $\lambda \in \Lambda_n$. Also, let m_λ be the monomial symmetric polynomial that corresponds to $\lambda \in \Lambda_n$. Recall that the usual order on Λ_n : for $\lambda, \mu \in \Lambda_n$ we say $\lambda \geq \mu$ if $\lambda_1 + \dots + \lambda_i \geq \mu_1 + \dots + \mu_i$ for all $i = 1, \dots, n$.

Proposition 3 [6, Theorem 2.4] For every $\lambda \in \Lambda_n$ there exists a unique symmetric polynomial $P_\lambda(z; q, t)$ in n variables such that $P_\lambda(\bar{\mu}; q, t) = 0$ for all $\mu \in \Lambda_n, |\mu| \leq |\lambda|, \mu \neq \lambda$, and which has an expansion $P_\lambda(z; q, t) = \sum_{\mu \leq \lambda} p_{\lambda\mu} m_\mu(z)$ with $p_{\lambda\lambda} = 1$.

Proposition 4 [6, Proposition 2.8] $P_\lambda(z; q, q) = q^{-(n-1)|\lambda|} \mathfrak{s}_\lambda(q^{n-1}z; q)$.

Now we can prove

Lemma 2 For any partition ν there exists a constant c_ν such that for all $\lambda \in \Lambda_n$

$$T_F(y_\nu)|_{\mathcal{H}_\lambda} = c_\nu \mathfrak{s}_\nu(q^{2(\lambda_1+n-1)}, q^{2(\lambda_2+n-2)}, \dots, q^{2(\lambda_{n-1}+1)}, q^{2\lambda_n}; q^2).$$

Proof. It follows from Theorem 2 and Lemma 1, that for an arbitrary partition ν there exists a symmetric polynomial of degree $|\nu|$ in n variables x_1, \dots, x_n , such that the eigenvalues $T_F(y_\nu)|_{\mathcal{H}_\lambda}$ are just the values of the polynomial at $x_1 = q^{2(\lambda_1+n-1)}, \dots, x_n = q^{2\lambda_n}$.

Let $\delta = (n-1, \dots, 1, 0)$, $q^{2(\mu+\delta)} \stackrel{\text{def}}{=} (q^{2(\mu_1+n-1)}, q^{2(\mu_2+n-2)}, \dots, q^{2(\mu_{n-1}+1)}, q^{2\mu_n})$ for any $\mu \in \Lambda_n$. Propositions 3 and 4 claim that $\mathfrak{s}_\nu(x_1, \dots, x_n; q^2)$ is a unique (up to a constant multiplier) symmetric polynomial of degree $|\nu|$ with $\mathfrak{s}_\nu(q^{2(\mu+\delta)}; q^2) = 0$ for all $\mu \in \Lambda_n$, $|\mu| \leq |\nu|$, $\mu \neq \nu$.

To finish the proof, one should investigate zeros of $T_F(y_\nu)$ to conclude the proof (cf. the proof in Sahi's paper [11]). We claim that

$$T_F(y_\nu)|_{\mathcal{H}_\lambda} = 0 \quad \text{for } |\nu| \leq |\lambda| \quad \text{unless } \nu = \lambda.$$

Indeed, it suffices to prove that $T_F(y_\nu)(v_\lambda f_0) = 0$ for partitions ν and λ such that $|\nu| \leq |\lambda|$, $\nu \neq \lambda$. Recall that $y_\nu = \sum v_j w_j$, where $\{v_j\} \subset \mathbb{C}[\text{Mat}_n]_{q,\nu}$ contains v_ν and $\{w_j\} \subset \mathbb{C}[\overline{\text{Mat}}_n]_{q,\nu}$ contains v_ν^* . It follows from the commutation relations (9) that $T_F(w_j)(v_\lambda f_0) = 0$ unless $w_j = v_\lambda^*$. So, $T_F(y_\nu)(v_\lambda f_0) = 0$ unless $\nu = \lambda$. \square

Introduce the notations: $\lambda - a\mathbf{1}^n = (\lambda_1 - a, \dots, \lambda_n - a)$. The next proposition completes the proof of Theorem 1.

Proposition 5 $c_\nu = (-q)^{|\nu|} q^{-\sum_{i=1}^n \nu_i(\nu_i+2n-2i)}$.

Proof. Let us compare $T_F(y_\nu)(v_\nu f_0)$ and $\mathfrak{s}_\nu(q^{2(\nu+\delta)}; q^2)$. By Proposition 6,

$$T_F(y_\nu)(v_\nu f_0) = (\det_q \mathbf{Z})^{\nu_n} T_F(y_{\nu-\nu_n \mathbf{1}^n}) T_F((\det_q \mathbf{Z})^*)^{\nu_n} v_\nu f_0.$$

By Theorem 2,

$$T_F((\det_q \mathbf{Z})^*)^{\nu_n} v_\nu f_0 = (-1)^{n\nu_n} q^{-n(n-1)\nu_n} \prod_{i=0}^{\nu_n-1} \mathfrak{s}_{\mathbf{1}^n}(q^{2(\nu+\delta-i\mathbf{1}^n)}; q^2) v_{\nu-\nu_n \mathbf{1}^n} f_0.$$

We proceed by induction in n . For $n = 1$ the statement follows from the last identity and Lemma 3 from the next section.

Let $n > 1$. One can rewrite some of the commutation relations (9) more explicitly:

$$\begin{aligned} (z_n^\alpha)^* z_b^\beta &= q \sum_{\alpha', \beta'=1}^n R(\alpha, \beta, \alpha', \beta') z_b^{\beta'} (z_n^{\alpha'})^* & \text{for } b < n, \\ (z_a^n)^* z_b^\beta &= q \sum_{a', b'=1}^n R(a, b, a', b') z_b^\beta (z_a^n)^* & \text{for } \beta < n. \end{aligned}$$

Hence, $T_F((z_a^\alpha)^*) v_{\nu-\nu_n \mathbf{1}^n} f_0 = 0$ for $a = n$ or $\alpha = n$. Denote by T'_F the faithful representation of $\text{Pol}(\text{Mat}_{n-1})_q$ in the vector space \mathcal{H}' defined by a single generator f'_0 and the relations $(z_a^\alpha)^* f'_0 = 0$, for $a, \alpha = 1, \dots, n-1$. Thus,

$$T_F(y_{\nu-\nu_n \mathbf{1}^n})|_{\mathcal{H}_{\nu-\nu_n \mathbf{1}^n}} = T'_F(y_\tau)|_{\mathcal{H}'_\tau},$$

where $\tau = (\nu_1 - \nu_n, \dots, \nu_{n-1} - \nu_n)$. Let $\delta' = (n-2, \dots, 2, 1, 0)$. Hence, by the inductive assumption,

$$T_F(y_{\nu-\nu_n \mathbf{1}^n}) v_{\nu-\nu_n \mathbf{1}^n} f_0 = (-q)^{|\tau|} q^{-\sum_{i=1}^{n-1} \tau_i(\tau_i+2n-2-2i)} \mathfrak{s}_\tau(q^{2(\tau+\delta')}; q^2) v_{\nu-\nu_n \mathbf{1}^n} f_0.$$

Now the required statement follows from Lemmas 3, 4 of the next section and the following computation

$$\begin{aligned}
T_F(y_\nu)v_\nu f_0 &= (-1)^{n\nu_n} q^{-n(n-1)\nu_n} \prod_{i=0}^{\nu_n-1} \mathfrak{s}_{1^n}(q^{2(\nu+\delta-i1^n)}; q^2) (\det_q \mathbf{z})^{\nu_n} T_F(y_{\nu-\nu_n \mathbf{1}^n}) v_{\nu-\nu_n \mathbf{1}^n} f_0 = \\
&= (-1)^{n\nu_n} q^{-n(n-1)\nu_n} \prod_{i=0}^{\nu_n-1} \mathfrak{s}_{1^n}(q^{2(\nu+\delta-i1^n)}; q^2) (-q)^{|\tau|} q^{-\sum_{i=1}^{n-1} \tau_i(\tau_i+2n-2-2i)} \mathfrak{s}_\tau(q^{2(\tau+\delta')}; q^2) v_\nu f_0 = \\
&= (-q)^{|\nu|} q^{-\sum_{i=1}^n \nu_i(\nu_i+2n-2i)} \mathfrak{s}_\nu(q^{2(\nu+\delta)}; q^2) v_\nu f_0. \square
\end{aligned}$$

Proposition 6 $y_\nu = (\det_q \mathbf{z})^{\nu_n} y_{\nu-\nu_n \mathbf{1}^n} ((\det_q \mathbf{z})^*)^{\nu_n}$.

Proof. It is obvious that $(\det_q \mathbf{z})^{\nu_n} y_{\nu-\nu_n \mathbf{1}^n} ((\det_q \mathbf{z})^*)^{\nu_n} \in \text{Pl}_\nu = \mathbb{C} \cdot y_\nu$, the statement follows from an explicit computation of the coefficient of $v_\nu v_\nu^*$. \square

3 q-factorial Schur functions

This section contains auxiliary statements which we used above. As usual, $(a)_n \stackrel{\text{def}}{=} \prod_{i=0}^{n-1} (a - q^{2i})$.

Lemma 3 For any partition $\nu \in \Lambda_n$, such that $\nu_n > 0$, one has

$$\mathfrak{s}_\nu(q^{2\nu+2\delta}; q^2) = q^{2|\nu|-2n} \mathfrak{s}_{1^n}(q^{2(\nu+\delta)}; q^2) \mathfrak{s}_{\nu-1^n}(q^{2(\nu+\delta-1^n)}; q^2).$$

Proof. The proof reduces to the explicit computation $\mathfrak{s}_\nu(q^{2\nu+2\delta}; q^2) =$

$$\begin{aligned}
&= \prod_{1 \leq i \leq j \leq n} \frac{1}{q^{2\nu_i+2n-2i} - q^{2\nu_j+2n-2j}} \begin{vmatrix} (q^{2\nu_1+2n-2})_{\nu_1+n-1} & 0 & \dots & 0 \\ (q^{2\nu_1+2n-2})_{\nu_2+n-2} & (q^{2\nu_2+2n-4})_{\nu_2+n-2} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ (q^{2\nu_1+2n-2})_{\nu_n} & (q^{2\nu_2+2n-4})_{\nu_n} & \dots & (q^{2\nu_n})_{\nu_n} \end{vmatrix} = \\
&= \prod_{i=1}^n (q^{2\nu_i+2n-2i} - 1) q^{2\nu_i+2n-4i-2} \prod_{1 \leq i \leq j \leq n} \frac{1}{q^{2\nu_i+2n-2i-2} - q^{2\nu_j+2n-2j-2}} \cdot \\
&\quad \cdot \begin{vmatrix} (q^{2\nu_1+2n-4})_{\nu_1+n-2} & 0 & \dots & 0 \\ (q^{2\nu_1+2n-4})_{\nu_2+n-3} & (q^{2\nu_2+2n-6})_{\nu_2+n-3} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ (q^{2\nu_1+2n-4})_{\nu_n-1} & (q^{2\nu_2+2n-6})_{\nu_n-1} & \dots & (q^{2\nu_n-2})_{\nu_n-1} \end{vmatrix} \\
&= q^{2|\nu|} q^{-2n} \mathfrak{s}_{1^n}(q^{2\nu+2\delta}; q^2) \mathfrak{s}_{\nu-1^n}(q^{2(\nu+\delta-1^n)}; q^2). \square
\end{aligned}$$

Lemma 4 For a partition $\nu \in \Lambda_n$ with $\nu_n = 0$ one has $\mathfrak{s}_\nu(q^{2(\nu+\delta)}; q^2) = q^{2|\nu|} \mathfrak{s}_{\nu'}(q^{2(\nu'+\delta')}; q^2)$, where $\nu' = (\nu_1, \dots, \nu_{n-1})$.

Proof. The proof is managed by the explicit computation

$$\begin{aligned}
\mathfrak{s}_\nu(q^{2(\nu+\delta)}; q^2) &= \prod_{1 \leq i \leq j \leq n} (q^{2\nu_i+2n-2i} - q^{2\nu_j+2n-2j})^{-1} \begin{vmatrix} (q^{2\nu_1+2n-2})_{\nu_1+n-1} & 0 & \dots & 0 \\ (q^{2\nu_1+2n-2})_{\nu_2+n-2} & (q^{2\nu_2+2n-4})_{\nu_2+n-2} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 1 & 1 & \dots & 1 \end{vmatrix} \\
&= \prod_{1 \leq i \leq j \leq n} (q^{2\nu_i+2n-2i} - q^{2\nu_j+2n-2j})^{-1} \begin{vmatrix} \frac{(q^{2\nu_1+2n-2})_{\nu_1+n-1}}{q^{2\nu_1+2n-2}-1} & 0 & \dots & 0 \\ \frac{(q^{2\nu_1+2n-2})_{\nu_2+n-2}}{q^{2\nu_1+2n-2}-1} & \frac{(q^{2\nu_2+2n-4})_{\nu_2+n-2}}{q^{2\nu_2+2n-4}-1} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ \frac{(q^{2\nu_1+2n-2})_{\nu_{n-1}}}{q^{2\nu_1+2n-2}-1} & \frac{(q^{2\nu_2+2n-4})_{\nu_{n-1}}}{q^{2\nu_2+2n-4}-1} & \dots & \frac{(q^{2\nu_{n-1}+2})_{\nu_{n-1}}}{q^{2\nu_{n-1}+2}-1} \end{vmatrix} \\
&= q^{2|\nu|} \mathfrak{s}_{\nu'}(q^{2\nu'+2\delta'}; q^2). \square
\end{aligned}$$

4 Acknowledgements

The author thanks to L. Vaksman for helping with the proof of Lemma 2 and constant attention to her work. Also thanks are due to D.Shklyarov. At last, the author would like to express her gratitude to a referee for many useful remarks.

References

- [1] O. Bershtein, Ye. Kolisnyk, L. Vaksman, *On a q -analog of the Wallach-Okounkov formula*, – Lett. in Math.Phys. 78 No. 1 (2006), pp.97-109.
- [2] L.Biedenharn, J.Louck, *A new class of symmetric polynomials defined in terms of tableaux*, – Adv. in Appl. Math. 10 (1989), pp.396-438.
- [3] J.C. Jantzen, *Lectures on Quantum Groups* – Amer. Math. Soc., Providence RI, (1996).
- [4] A. Klimyk, K. Schmüdgen, *Quantum Groups and Their Representations* – Springer, Berlin, (1997).
- [5] F. Knop, S. Sahi, *Difference equations and symmetric polynomials defined by their zeros*, – IMRN 10 (2000), pp.473-486.
- [6] F. Knop, *Symmetric and non-symmetric quantum Capelli polynomials*, – Comment. Math. Helv. 72 (1997), pp.84-100.
- [7] B.Kostant, S.Sahi, *The Capelli identity, tube domains and the generalized Laplace transform*, – Adv.Math. 87 (1991), pp.71-92.
- [8] B.Kostant, S.Sahi, *Jordan algebras and Capelli identities*, – Inv.Math. 112 (1993), pp.657-664.
- [9] I. Macdonald, *Schur functions: theme and variations*, – I.R.M.A. Strasbourg (1992) 498/S-27, pp.5-39.
- [10] S.Sahi, *Interpolation, integrality and a generalization of Macdonald's polynomials*, – Int.Math.Res.Not. 10 (1996), pp.457-471.

- [11] S. Sahi, *The Spectrum of certain invariant differential operators associated to a Hermitian symmetric space*, – in Lie Theory and Geometry, ed. J.-L. Brylinsky, R. Brylinsky, V. Guillemin, V. Kac (1994), pp. 569-576.
- [12] D. Shklyarov, S. Sinel'shchikov, L. Vaksman, *q-Analogs of some bounded symmetric domains*, – Czech. J. of Phys. 50 No.1 (2000), pp. 175-180.
- [13] D. Shklyarov, S. Sinel'shchikov, L. Vaksman, *Fock representations and quantum matrices*, – International J. Math 15 No.9 (2000), pp.1-40.